Simulation of Antenna Arrays: Part 2 – Theory of the Array Factor

In this blog we will outline some of the theoretical considerations for analyzing arrays of antennas using array factors. For those of you who are squeamish there is a lot of mathematics to get through! Depending on the performance requirements, the number of antennas in an array can be as small as 2, or as large as several thousand; in general, the performance of an antenna array increases with the number of elements in the array. Elements of the array can be arranged in various configurations, such as planar, circular, etc. Planar arrays are versatile, provide symmetrical patterns with lower side lobes, high directivity (narrow main beam) and can be used to scan the main beam of the antenna towards any point in space. Consequently, applications for planar arrays include tracking and search radar, remote sensing, communications, etc.

To calculate the array factor, we need to consider the radiation fields developed when multiple antenna operate simultaneously. The radiation field of a single antenna is characterized by the radiation vector:

$$\boldsymbol{F}(\boldsymbol{k}) = \int \boldsymbol{J}(\boldsymbol{r}) e^{j\boldsymbol{k}\cdot\boldsymbol{r}} d^3\boldsymbol{r}$$

where $\boldsymbol{J}(\boldsymbol{r})$ is the current density of the antenna and $k = 2\pi/\lambda$ is the wavenumber.

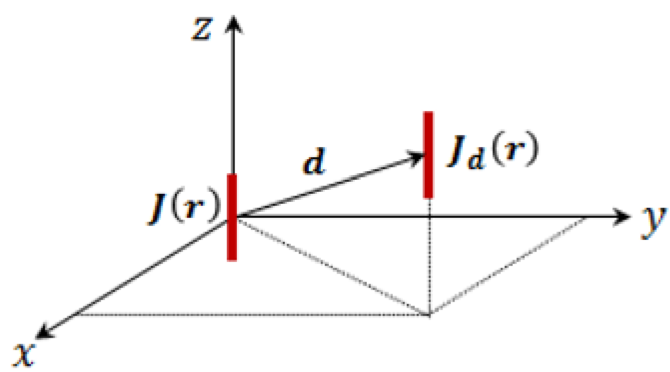

Figure 2: Antenna translated by **d**

Consider an antenna translated by the vector $\boldsymbol{d}$, as shown in Figure 2. The current density of the translated antenna will be $\boldsymbol{J_d}(\boldsymbol{r}) = \boldsymbol{J}(\boldsymbol{r} - \boldsymbol{d})$. The radiation vector of the translated antenna is then given by:

$$\boldsymbol{F_d} = \int e^{j\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{J}(\boldsymbol{r} - \boldsymbol{d}) d^3\boldsymbol{r} = \int e^{j\boldsymbol{k}\cdot(\boldsymbol{r}' - \boldsymbol{d})} \boldsymbol{J}(\boldsymbol{r}') d^3\boldsymbol{r}' = e^{j\boldsymbol{k}\cdot\boldsymbol{d}} \boldsymbol{F}$$

where $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{d}$.

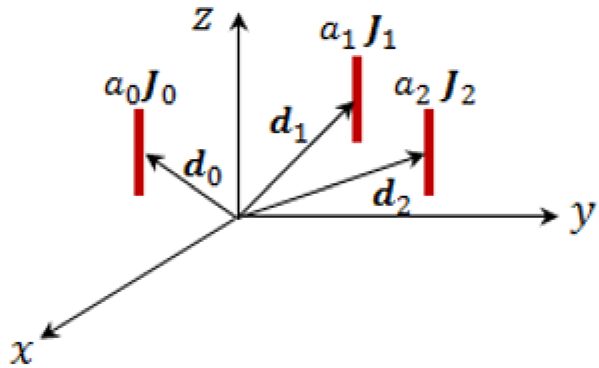

Figure 3: Three-dimensional array of multiple identical antenna

Consider a three-dimensional array of identical antennas located at positions $\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2, \cdots$ with relative feed coefficients $a_0, a_1, a_2, \cdots$, Figure 3.

The current density of the n −th antenna will be $\boldsymbol{J}_n(\boldsymbol{r}) = a_n \boldsymbol{J}(\boldsymbol{r} - \boldsymbol{d}_n)$ and the corresponding radiation vector:

$$\boldsymbol{F}_n(\boldsymbol{k}) = a_n e^{j\boldsymbol{k}\cdot\boldsymbol{d}_n} \boldsymbol{F}(\boldsymbol{k})$$

The total current density of the array is:

$$\boldsymbol{J}_{tot} = a_0 \boldsymbol{J}(\boldsymbol{r} - \boldsymbol{d}_0) + a_1 \boldsymbol{J}(\boldsymbol{r} - \boldsymbol{d}_1) + a_2 \boldsymbol{J}(\boldsymbol{r} - \boldsymbol{d}_2) + \cdots$$

and the total array radiation vector:

$$\boldsymbol{F}_{tot}(\boldsymbol{k}) = \boldsymbol{F}_0 + \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots = a_0 e^{j\boldsymbol{k}\cdot\boldsymbol{d}_0} \boldsymbol{F}(\boldsymbol{k}) + a_1 e^{j\boldsymbol{k}\cdot\boldsymbol{d}_1} \boldsymbol{F}(\boldsymbol{k}) + a_2 e^{j\boldsymbol{k}\cdot\boldsymbol{d}_2} \boldsymbol{F}(\boldsymbol{k}) + \cdots$$

The factor $\boldsymbol{F}(\boldsymbol{k})$ due to a single antenna element at the coordinate origin is common to all terms, thus:

$$\boldsymbol{F}_{tot}(\boldsymbol{k}) = A(\boldsymbol{k}) \boldsymbol{F}(\boldsymbol{k})$$

where $A(\boldsymbol{k})$ is the array factor given by:

$$A(\boldsymbol{k}) = a_0 e^{j\boldsymbol{k}\cdot\boldsymbol{d}_0} + a_1 e^{j\boldsymbol{k}\cdot\boldsymbol{d}_1} + a_2 e^{j\boldsymbol{k}\cdot\boldsymbol{d}_2} + \cdots$$

The net effect of an array of identical antennas is to modify the single-antenna radiation vector by the array factor $A(\boldsymbol{k})$, this incorporates all the translational phase shifts and relative feed coefficients of the array elements.
Consider uniformly-spaced one-dimensional array of N antennas along the x −axis, as shown in Figure 4.

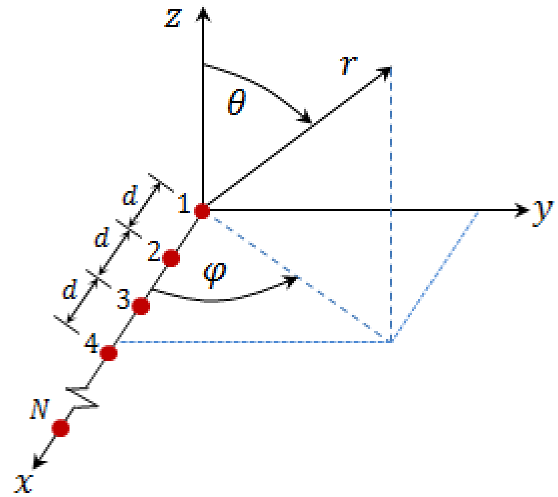

Figure 4: Geometry of linear array

The individual elements are positioned at locations $x_n, n = 0,1,2,\cdots,$ with displacement vectors $\boldsymbol{d}_n = x_n\hat{\boldsymbol{x}}$. The associated array factor is:

$$A(\theta,\varphi) = \sum_{n=0}^{N-1} a_n e^{j\boldsymbol{k}\cdot\boldsymbol{d}_n} = \sum_{n=0}^{N-1} a_n e^{jk_x x_n} = \sum_{n=0}^{N-1} a_n e^{jkx_n sin\theta cos\varphi}$$

where $k_x = ksin\theta cos\varphi$.

For an array of equally-spaced antenna, the element locations are $x_n = nd$, where d is the distance between elements. Assuming all the elements have identical excitation amplitudes and each element is excited with a linear phase progression, then:

$$a_0 = 1, a_1 = e^{j\alpha}, a_2 = e^{j2\alpha}, \cdots$$

In this case, the array factor becomes:

$$A(\theta,\varphi) = 1 + e^{j\psi} + e^{j2\psi} + \cdots + e^{j(N-1)\psi}$$

Where the digital wavenumber variable $\psi = kdsin\theta cos\varphi + \alpha$.

This can be rewritten as:

$$A(\theta,\varphi)e^{j\psi} = e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots + e^{j(N-1)\psi} + e^{jN\psi}$$

So that we obtain:

$$A(\theta,\varphi)\left(1 - e^{j\psi}\right) = 1 - e^{jN\psi}$$

Thus, the linear array factor is calculated as:

$$A(\theta,\varphi) = \frac{1 - e^{jN\psi}}{1 - e^{j\psi}} = \frac{e^{jN\psi/2}\left(e^{-jN\psi/2} - e^{jN\psi/2}\right)}{e^{j\psi/2}\left(e^{-j\psi/2} - e^{j\psi/2}\right)} = e^{j(N-1)\psi/2}\frac{sin(N\psi/2)}{sin(\psi/2)}$$

If the reference point is the physical center of the linear array, the front phase multiplication factor is unity and thus:

$$A(\theta,\varphi) = \frac{sin(N\psi/2)}{sin(\psi/2)}$$

The maximum value of A is equal to N. To normalize the array factors so that the maximum values of each is equal to unity, the radiation pattern is generally normalized with respect to the number of elements in the array and the array factor for a linear array is:

$$A(\theta,\varphi) = \frac{1}{N}\frac{sin(N\psi/2)}{sin(\psi/2)}$$

For the planar array shown in Figure 5,

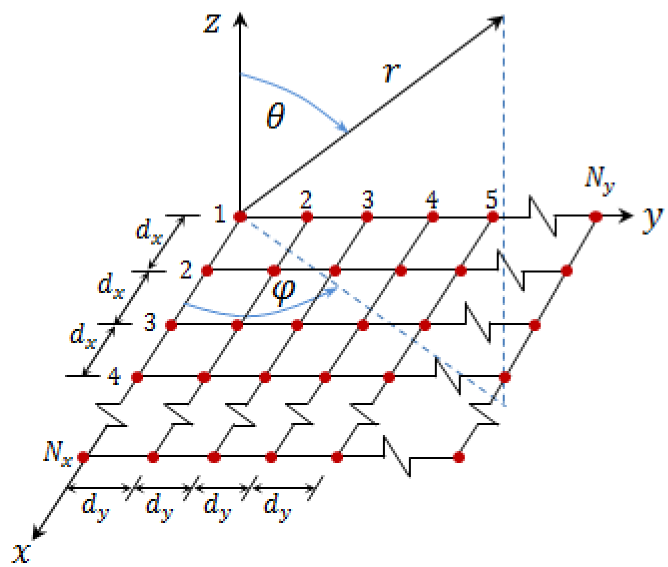

Figure 5: Planar array geometry

The array factor is given by:

$$A(\theta,\varphi) = \frac{1}{N_x}\frac{sin\left[\frac{N_x}{2}\left(2\pi\frac{d_x}{\lambda}sin\theta\ cos\varphi + \alpha_x\right)\right]}{sin\left[\frac{1}{2}\left(2\pi\frac{d_x}{\lambda}sin\theta\ cos\varphi + \alpha_x\right)\right]} \times \frac{1}{N_y}\frac{sin\left[\frac{N_y}{2}\left(2\pi\frac{d_y}{\lambda}sin\theta\ cos\varphi + \alpha_y\right)\right]}{sin\left[\frac{1}{2}\left(2\pi\frac{d_y}{\lambda}sin\theta\ cos\varphi + \alpha_y\right)\right]}$$

where N_x is the number of x −elements separated by distance d_x, N_y is the number of y −elements separated by distance d_y, α_x is the linear phase progression excitation of x −elements, α_{xy} is the linear phase progression excitation of y −elements, θ is the elevation angle, φ is the azimuth angle, and λ is the wavelength in free-space.

Extension to the case of three-dimensional arrays is intuitively obvious but less practical.